PART A: Answer only three of the four questions below.

A 1 Show that if \( G \) is a non-abelian simple subgroup of \( S_n \) then \( G \) is contained in \( A_n \).

A 2 Let \( R \) be a commutative ring with 1 in which every ideal is prime. Prove that \( R \) is a field.

A 3 Let \( R \) be a ring and \( f : \mathbb{Q} \to R \) and \( g : \mathbb{Q} \to R \) be ring homomorphisms. Show that if \( f|_\mathbb{Z} = g|_\mathbb{Z} \) then \( f = g \).

A 4 Let \( W \) be the space of \( n \times n \)-matrices over a field \( F \) and let \( f \) be a linear functional on \( W \) such that \( f(AB) = f(BA) \) for every \( A, B \in W \). Show that \( f \) is a multiple of the trace functional.

PART B: Answer only three of the four questions below.

B 1 Let \((M,d)\) be a metric space, \( A \subset M \) be nonempty, \( x \in A \), and \( B(x,r) = \{y \in M : d(x,y) < r\} \) for every \( r > 0 \). Prove or disprove each of the following statements.
   (a) If \( A \) is closed and \( A \subset B(x,r) \) for some \( r > 0 \), then \( A \) is compact.
   (b) If \( A \) is compact, then \( A \) is closed and \( A \subset B(x,r) \) for some \( r > 0 \).

B 2 Let \( f : \mathbb{C} \to \mathbb{C} \) be an entire function (i.e., \( f \) is analytic on the whole complex plane).
   (a) Suppose \( f \) is bounded by a constant \( M \) on the circle \( \{z \in \mathbb{C} : |z| = R\} \) for some \( R > 0 \). Prove that the coefficients \( C_k \) in the power series expansion of \( f \) about 0 satisfy
      \[ |C_k| \leq \frac{M}{R^k}. \]
   (b) Suppose there exist real constants \( A, B \) and an integer \( n \geq 0 \) such that \( |f(z)| \leq A + B|z|^n \) for every \( z \in \mathbb{C} \). Prove that \( f \) is a polynomial of degree at most \( n \). (Hint: Use part (a).)

B 3 Let \( m \) denote the Lebesgue measure on the real line, \( f : \mathbb{R} \to \mathbb{R} \) be an integrable function and \( F(x) = \int_{-\infty}^{x} f \, dm \) for every \( x \in \mathbb{R} \). Prove or disprove each of the following statements. Indicate the theorems you use (if any).
   (a) \( F \) is continuous at every \( x \in \mathbb{R} \).
   (b) \( F \) is differentiable at every \( x \in \mathbb{R} \).
   (c) \( F \) is differentiable at \( \mu \)-a.e. \( x \in \mathbb{R} \).

B 4 Let \((X,\mathcal{F},\mu)\) be a measure space and \((f_n)_{n \geq 1} \) be a sequence of real-valued measurable functions on \( X \). Prove or disprove each of the following statements.
   (a) If \( f_n \to 0 \) in \( \mu \)-measure, then \( f_n \to 0 \) \( \mu \)-a.e.
   (b) If \( f_n \to 0 \) \( \mu \)-a.e., then \( f_n \to 0 \) in \( \mu \)-measure.
   (c) If \( \mu(X) < \infty \) and \( f_n \to 0 \) \( \mu \)-a.e., then \( f_n \to 0 \) in \( \mu \)-measure.